

Nonlinear-damping continuation of the nonlinear Schrödinger equation – a numerical study

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Abstract

We study the nonlinear-damping continuation of singular solutions of the critical and supercritical NLS. Our simulations suggest that for generic initial conditions that lead to collapse in the undamped NLS, the solution of the weakly-damped NLS

$$i\psi_t(t, \mathbf{x}) + \Delta\psi + |\psi|^{p-1}\psi + i\delta|\psi|^{q-1}\psi = 0, \quad 0 < \delta \ll 1,$$

is highly asymmetric with respect to the singularity time, and the post-collapse defocusing velocity of the singular core goes to infinity as the damping coefficient δ goes to zero. In the special case of the minimal-power blowup solutions of the critical NLS, the continuation is a minimal-power solution with a higher (but finite) defocusing velocity, whose magnitude increases monotonically with the nonlinear damping exponent q .

1. Introduction

The nonlinear Schrödinger equation (NLS)

$$i\psi_t(t, \mathbf{x}) + \Delta\psi + |\psi|^{p-1}\psi = 0, \quad \psi_0(0, \mathbf{x}) = \psi_0(\mathbf{x}) \in H^1, \quad (1)$$

where $\mathbf{x} = (x_1, \dots, x_d) \in \mathbb{R}^d$ and $\Delta = \partial_{x_1 x_1} + \dots + \partial_{x_d x_d}$, is one of the canonical nonlinear equations in physics, arising in various fields such as nonlinear optics, plasma physics, Bose-Einstein condensates (BEC), and surface waves. When $(p-1)d < 4$, the NLS is called subcritical. In that case, all H^1 solutions exist globally. In contrast, both the critical NLS $(p-1)d = 4$ and the supercritical NLS $(p-1)d > 4$ admit singular solutions. Since physical quantities do not become singular, this implies that some of the terms that were neglected in the derivation of the NLS, become important near the singularity.

The continuation of NLS solutions beyond the singularity has been an open question for many years. In 1992, Merle [1] presented a continuation of the explicit blowup solutions $\psi_{\text{explicit}, \alpha}$ of the critical NLS, see (9), which is based on slightly reducing the power (L^2 norm) of the initial condition. This continuation has two key properties:

1. **Property 1:** The solution is symmetric with respect to the singularity time T_c .
2. **Property 2:** After the singularity, the solution can only be determined up to multiplication by a constant phase term $e^{i\theta}$.

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More recently, Merle, Raphael and Szeftel [2] generalized this continuation result to Bourgain-Wang singular solutions [3]. Note, however, that both the explicit solutions $\psi_{\text{explicit},\alpha}$ and the Bourgain-Wang solutions are unstable.

In [4], Merle presented a different continuation, which is based on the addition of nonlinear saturation. Merle showed that, generically, as the nonlinear saturation coefficient goes to zero, the limiting solution beyond T_c can be decomposed into two components: A δ -function singular core that extends for $T_c \leq t \leq T^0$, and a regular component elsewhere.

In [5], Tao proved the global existence and uniqueness in the semi Strichartz class for solutions of the critical NLS. Intuitively, these solutions are formed by solving the equation in the Strichartz class whenever possible, and deleting any power that escapes to spatial or frequency infinity when the solution leaves the Strichartz class. These solutions, however, do not depend continuously on the initial conditions, and are thus not a well-posed class of solutions. Recently, Stinis [6] studied numerically the continuation of singular NLS solutions using the t-model approach.

In [7] we analyzed asymptotically and numerically four potential continuations of singular NLS solutions: 1) a sub-threshold power continuation, 2) a shrinking-hole continuation for ring-type solutions, 3) a vanishing nonlinear-damping continuation, and 4) a complex Ginzburg-Landau (CGL) continuation. Our main findings were as follows:

1. The non-uniqueness of the phase of the singular core beyond the singularity (Property 2) is a universal feature of NLS continuations.
2. The symmetry with respect to the singularity time (Property 1) holds if the continuation model is time reversible and if it leads to a point singularity (i.e., if it defocuses for $t > T_c$). Therefore, it is a non-generic feature.

Recently, the post-collapse loss-of-phase phenomena was demonstrated experimentally for intense laser beams propagating in water [8].

In this paper we further study the effect of small nonlinear-damping in the NLS

$$i\psi_t(t, \mathbf{x}) + \Delta\psi + |\psi|^{p-1}\psi + i\delta|\psi|^{q-1}\psi = 0, \quad 0 < \delta \ll 1. \quad (2)$$

The addition of small nonlinear-damping is physical. Indeed, in nonlinear optics, experiments suggest that arrest of collapse is related to plasma formation, and nonlinear damping is used as phenomenological model for multi-photon absorption by plasma. In BEC, a quintic nonlinear damping term corresponds to losses from condensate due to three-body inelastic recombinations [9]. In addition, the nonlinear-damping term appears in the complex-Ginzburg-Landau (CGL) equation, which arises in a models of chemical turbulence, Poiseuille flow, Rayleigh-Bérnard convection, Taylor-Couette flow, and superconductivity,

In [7] we analyzed the continuation of the critical NLS with a vanishing critical nonlinear damping, i.e., equation (2) with $p = q = 1 + 4/d$. Since the NLS (2) is not time reversible, its solutions are asymmetric with respect to the time $T_{\text{arrest}}^{(\delta)}$ at which the collapse is arrested. In particular, in the limit $\delta_n \rightarrow 0+$, the continuation of $\psi_{\text{explicit},\alpha}(t, r)$ is $e^{i\theta}\psi_{\text{explicit},\kappa\alpha}^*(2T_c - t, r)$, where $\kappa \approx 1.614$. Hence, the defocusing velocity $\kappa\alpha$ is higher than the focusing velocity α . When the initial condition leads to a loglog collapse in the undamped critical NLS, asymptotic analysis and numerical simulations suggest that the singular core expands beyond the singularity at a velocity that goes to infinity as $\delta \rightarrow 0+$.

The question that we address in this study is whether and how the results of [7] for $q = p = 1 + 4/d$ will change in the following cases:

1. The critical NLS with a supercritical damping exponent (i.e., $q > p = 1 + 4/d$).

2. The supercritical NLS with $q \geq p > 1 + 4/d$.

The paper is organized as follows. In Section 2 we provide a short review of NLS theory. In Section 3 we review previous rigorous, asymptotic, and numerical results on the effect of damping in the NLS. In Section 4 we show numerically that in the supercritical NLS, the nonlinear damping exponent q has to be strictly higher than the nonlinearity exponent p , in order to arrest the collapse. This is different from the critical case, where collapse is arrested for $q \geq p$. In Section 5 we show that solutions of the supercritical NLS with a small nonlinear damping are asymmetric with respect to the arrest-of-collapse time $T_{\text{arrest}}^{(\delta)}$, and that the post-collapse defocusing velocity of the singular core goes to infinity as the damping coefficient δ goes to zero. In Section 6 we obtain similar results for the critical NLS with generic initial conditions that lead to a loglog collapse. In the special case of the minimal-power explicit blowup solution $\psi_{\text{explicit},\alpha}(t, r)$ of the critical NLS, however, the continuation beyond the singularity is also defined for $q < p$, and is given by $e^{i\theta} \psi_{\text{explicit},\kappa(q)\alpha}^*(2T_c - t, r)$, where $\kappa(q)$ increases monotonically with q . Final remarks are given in Section 7.

Overall, the qualitative effect of small nonlinear damping on the collapse is the same in the critical and supercritical NLS. One difference is that in the critical case collapse is arrested for $q \geq p$, whereas in the supercritical case collapse is only arrested for $q > p$. Another difference is that the distance between the damped solution around $T_{\text{arrest}}^{(\delta)}$ and the asymptotic profile of the undamped NLS is small in the critical case, but large in the supercritical case. Surprisingly, in the latter case, the profile near $T_{\text{arrest}}^{(\delta)}$ appears to be given by a rescaled supercritical standing wave.

2. Review of NLS theory

The NLS (1) has two important conservation laws: *Power* conservation¹

$$P(t) \equiv P(0), \quad P(t) = \int |\psi|^2 d\mathbf{x},$$

and *Hamiltonian* conservation

$$H(t) \equiv H(0), \quad H(t) = \int |\nabla \psi|^2 d\mathbf{x} - \frac{2}{p+1} \int |\psi|^{p+1} d\mathbf{x}. \quad (3)$$

The NLS (1) admits the waveguide solutions $\psi = e^{it} R(r)$, where $r = |\mathbf{x}|$, and R is the solution of

$$R''(r) + \frac{d-1}{r} R' - R + R^p = 0, \quad R'(0) = 0, \quad R(\infty) = 0. \quad (4)$$

When $d = 1$, the solution of (4) is unique, and is given by

$$R_p(x) = \left(\frac{p+1}{2} \right)^{1/(p-1)} \cosh^{-2/(p-1)} \left(\frac{p-1}{2} x \right). \quad (5)$$

When $d \geq 2$, equation (4) admits an infinite number of solutions. The solution with the minimal power, which we denote by $R^{(0)}$, is unique, and is called the ground state.

¹We call the L^2 norm the *power*, since in optics it corresponds to the beam's power.

2.1. Critical NLS

In the critical case $(p-1)d = 4$, equation (1) can be rewritten as

$$i\psi_t(t, \mathbf{x}) + \Delta\psi + |\psi|^{4/d}\psi = 0, \quad \psi_0(0, \mathbf{x}) = \psi_0(\mathbf{x}) \in H^1, \quad (6)$$

and equation (4) can be rewritten as

$$R''(r) + \frac{d-1}{r}R' - R + R^{4/d+1} = 0, \quad R'(0) = 0, \quad R(\infty) = 0. \quad (7)$$

Theorem 1 (Weinstein [10]). *A sufficient condition for global existence in the critical NLS (6) is $\|\psi_0\|_2^2 < P_{\text{cr}}$, where $P_{\text{cr}} = \|R^{(0)}\|_2^2$, and $R^{(0)}$ is the ground state of equation (7).*

The critical NLS (6) admits the explicit solution

$$\psi_{\text{explicit}}(t, r) = \frac{1}{L^{d/2}(t)} R^{(0)} \left(\frac{r}{L(t)} \right) e^{i\tau + i\frac{L_t}{L} \frac{r^2}{4}}, \quad (8a)$$

where

$$L(t) = T_c - t, \quad \tau(t) = \int_0^t \frac{1}{L^2(s)} ds = \frac{1}{T_c - t}. \quad (8b)$$

More generally, applying the dilation transformation with $\lambda = \alpha$ and the temporal translation $T_c \rightarrow \alpha^2 T_c$ shows that the critical NLS (6) admits the explicit solutions

$$\psi_{\text{explicit}, \alpha}(t, r) = \frac{1}{L_\alpha^{d/2}(t)} R^{(0)} \left(\frac{r}{L_\alpha(t)} \right) e^{i\tau_\alpha + i\frac{(L_\alpha)_t}{L_\alpha} \frac{r^2}{4}}, \quad (9a)$$

where

$$L_\alpha(t) = \alpha(T_c - t), \quad \tau_\alpha(t) = \int_0^t \frac{1}{L_\alpha^2(s)} ds = \frac{1}{\alpha^2} \frac{1}{T_c - t}, \quad \alpha > 0. \quad (9b)$$

The explicit solutions (8)–(9) become singular at $t = T_c$. These solutions are unstable, however, as they have exactly the critical power for collapse. Therefore, any infinitesimal perturbation which decreases their power, will arrest the collapse.

When a solution of the critical NLS, whose power is slightly above P_{cr} , undergoes a stable collapse, it splits into two components: A collapsing core that approaches the universal ψ_R profile and blows up at the loglog law rate, and a non-collapsing tail (ϕ) that does not participate in the collapse process:

Theorem 2 (Merle and Raphael [11], [12], [13], [14], [15], [16], [17]). *Let $d = 1, 2, 3, 4, 5$, and let ψ be a solution of the critical NLS (6) that becomes singular at T_c . Then, there exists a universal constant $m^* > 0$, which depends only on the dimension, such that for any $\psi_0 \in H^1$ such that*

$$P_{\text{cr}} < \|\psi_0\|_2^2 < P_{\text{cr}} + m^*, \quad H_G(\psi_0) := H(\psi_0) - \left(\frac{\text{Im} \int \psi_0^* \nabla \psi_0}{\|\psi_0\|_2} \right)^2 < 0,$$

the following hold:

1. *There exist parameters $(\tau(t), \mathbf{x}_0(t), L(t)) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^+$, and a function $0 \neq \phi \in L^2$, such that*

$$\psi(t, \mathbf{x}) - \psi_R(t, \mathbf{x} - \mathbf{x}_0(t)) \xrightarrow{L^2} \phi(\mathbf{x}), \quad t \rightarrow T_c,$$

where

$$\psi_R(t, \mathbf{x}) = \frac{1}{L^{d/2}(t)} R^{(0)} \left(\frac{|\mathbf{x}|}{L(t)} \right) e^{i\tau(t)}, \quad (10)$$

and $R^{(0)}$ is the ground state of equation (7).

2. As $t \rightarrow T_c$,

$$L(t) \sim \sqrt{2\pi} \left(\frac{T_c - t}{\log |\log(T_c - t)|} \right)^{1/2} \quad (\text{loglog law}). \quad (11)$$

2.2. Supercritical NLS

In contrast to the extensive theory on singularity formation in the critical NLS, much less is known about the supercritical case

$$i\psi_t(t, \mathbf{x}) + \Delta\psi + |\psi|^{p-1}\psi = 0, \quad (p-1)d > 4. \quad (12)$$

Numerical simulations and formal calculations (see [18, Chapter 7] and the references therein), and recent rigorous analysis in the slightly-supercritical regime $0 < \frac{(p-1)d}{2} - 2 \ll 1$ [19] show that peak-type singular solutions of the supercritical NLS (12) collapse with a self-similar asymptotic profile ψ_Q , where

$$\psi_Q(t, r) = \frac{1}{L^{2/(p-1)}(t)} Q(\rho) e^{i\tau + i\frac{L_t}{L}r^2}, \quad \rho = \frac{r}{L(t)}, \quad \tau = \int_0^t \frac{ds}{L^2(s)}. \quad (13)$$

The blowup rate of $L(t)$ is a square root, i.e.,

$$L(t) \sim \kappa \sqrt{T_c - t}, \quad t \rightarrow T_c, \quad (14)$$

where $\kappa > 0$. In addition, the self-similar profile Q is the zero-Hamiltonian, monotonically-decreasing solution of

$$Q''(\rho) - \left(1 + i\frac{p-5}{4(p-1)}\kappa^2 - \frac{\kappa^4}{16}\rho^2 \right) Q + |Q|^{p-1}Q = 0, \quad Q'(0) = 0. \quad (15)$$

3. Effect of linear and nonlinear damping - review

In [20], Fibich studied asymptotically and numerically the effect of damping on blowup in the critical NLS. He showed that when the damping is linear, i.e.,

$$i\psi_t(t, \mathbf{x}) + \Delta\psi + |\psi|^{4/d}\psi + i\delta\psi = 0, \quad \psi(0, \mathbf{x}) = \psi_0(\mathbf{x}), \quad (16)$$

if the initial condition $\psi_0(\mathbf{x})$ is such that the solution of (16) becomes singular for $\delta = 0$, then the solution of (16) exists globally only if δ is above a threshold value $\delta_c > 0$ (which depends on ψ_0). Therefore, *linear damping cannot play the role of "viscosity" in continuations of solutions of the NLS*. When, however, the damping exponent is critical or supercritical, i.e.,

$$i\psi_t(t, \mathbf{x}) + \Delta\psi + (1 + i\delta) |\psi|^{4/d}\psi = 0, \quad 0 < \delta \ll 1, \quad (17)$$

or

$$i\psi_t(t, \mathbf{x}) + \Delta\psi + |\psi|^{4/d}\psi + i\delta|\psi|^{q-1}\psi = 0, \quad 0 < \delta \ll 1 \quad q-1 > 4/d, \quad (18)$$

respectively, then regardless of how small δ is, collapse is always arrested. Therefore, Fibich suggested that nonlinear damping can "play the role of viscosity" in defining weak NLS solutions, i.e., we can define the continuation

$$\psi := \lim_{\delta \rightarrow 0+} \psi^{(\delta)}, \quad (19)$$

where $\psi^{(\delta)}$ is the solution of (17) or (18).

Passot, Sulem and Sulem proved that high-order nonlinear damping always prevents collapse for $d = 2$. Antonelli and Sparber extended this result to $d = 1$ and $d = 3$:

Theorem 3 ([21, 22]). *The d -dimensional cubic NLS with nonlinear damping*

$$i\psi_t + \Delta\psi + \lambda|\psi|^2\psi + i\delta|\psi|^{q-1}\psi = 0, \quad \lambda \in \mathbb{R}, \quad \delta > 0, \quad (20)$$

where $\psi_0(\mathbf{x}) \in H^1(\mathbb{R}^d)$, $3 < q < \infty$ if $d = 1, 2$, and $3 < q < 5$ if $d = 3$, has a unique global in-time solution.

This rigorously shows that high-order nonlinear damping can play the role of "viscosity". More recently, Antonelli and Sparber proved global existence for the case where the damping exponent is equal to that of the nonlinearity:

Theorem 4 ([22]). *Consider the cubic nonlinear NLS with a cubic nonlinear damping*

$$i\psi_t(t, \mathbf{x}) + \Delta\psi + (1 + i\delta)|\psi|^2\psi = 0, \quad (21)$$

where $\psi_0(\mathbf{x}) \in H^1(\mathbb{R}^d)$, $\mathbf{x}\psi_0 \in L^2(\mathbb{R}^d)$, and $d \leq 3$. Then, for any $\delta \geq 1$, equation (21) has a unique global in-time solution.

Theorem 4 does not show that critical nonlinear damping can play the role of viscosity. We note, however, that the asymptotic analysis and simulations of [7, 20] strongly suggest that the solution of (17) exists globally for any $0 < \delta \ll 1$.

3.1. Explicit continuation of ψ_{explicit}

In [7], Fibich and Klein calculated explicitly the vanishing nonlinear-damping limit (19) of the explicit solution ψ_{explicit} :

Continuation Result 1 ([7]). *Let $\psi^{(\delta)}(t, r)$ be the solution of the NLS (17) with the initial condition*

$$\psi_0(r) = \psi_{\text{explicit}}(0, r), \quad (22)$$

see (8). Then, for any $\theta \in \mathbb{R}$, there exists a sequence $\delta_n \rightarrow 0+$ (depending on θ), such that

$$\lim_{\delta_n \rightarrow 0+} \psi^{(\delta_n)}(t, r) = \begin{cases} \psi_{\text{explicit}}(t, r) & 0 \leq t < T_c, \\ \psi_{\text{explicit}, \kappa}^*(2T_c - t, r)e^{i\theta} & T_c < t < \infty, \end{cases} \quad (23)$$

where $\psi_{\text{explicit}, \kappa}$ is given by (9a) with $\alpha = \kappa$,

$$\kappa = \pi [B_i(0)A_i'(s^*) - A_i(0)B_i'(s^*)] \approx 1.614, \quad (24)$$

$A_i(s)$ and $B_i(s)$ are the Airy and Bairy functions, respectively, and $s^* \approx -2.6663$ is the first negative root of $G(s) = \sqrt{3}A_i(s) - B_i(s)$.

In particular, the limiting width of the solution is given by

$$\lim_{\delta \rightarrow 0+} L(t) = \begin{cases} T_c - t & 0 \leq t < T_c, \\ \kappa(t - T_c) & T_c < t < \infty, \end{cases} \quad (25)$$

Therefore, the continuation is also an explicit minimal-power solution, but with a higher (defocusing) velocity. In addition, the solution beyond T_c is only determined up to multiplication by an unknown constant phase $e^{i\theta}$.

3.2. Continuation for loglog collapse

In [7], Fibich and Klein showed asymptotically and numerically that the continuation of solutions that undergo a loglog collapse has an infinite-velocity expanding core, that is determined up to a multiplicative constant phase $e^{i\theta}$:

Continuation Result 2 ([7]). *Let $\psi_0(r)$ be a radial initial condition, such that the corresponding solution ψ of the undamped critical NLS (6) collapses with the ψ_R profile at the loglog law blowup rate at T_c . Let $\psi^{(\delta)}$ be the solution of the damped NLS (17) with the same initial condition. Then,*

$$\lim_{\delta \rightarrow 0+} \psi^{(\delta)} = \psi, \quad 0 \leq t < T_c.$$

In addition, for any $0 < \delta \ll 1$, there exists $\theta(\delta) \in \mathbb{R}$, and a function $\phi \in L^2$, such that

$$\lim_{\delta \rightarrow 0+} \left[\psi^{(\delta)}(t, r) - \psi_R^*(2T_c - t, r; \delta) e^{i\theta(\delta)} \right] \xrightarrow{L^2} \phi(r), \quad t \longrightarrow T_c+,$$

where ψ_R is given by (10) with some function $L(t; \delta)$, such that

$$\lim_{t \rightarrow T_c+} \lim_{\delta \rightarrow 0+} L(t; \delta) = 0, \quad \lim_{t \rightarrow T_c+} \lim_{\delta \rightarrow 0+} L_t(t; \delta) = \infty, \quad \lim_{\delta \rightarrow 0+} \theta(\delta) = \infty.$$

Remark 1. We use the terminology Continuation Result, in order to emphasize that the proofs of Continuation Results 1 and 2 are based on asymptotic analysis and numerical simulations, and are not rigorous.

4. The critical damping exponent

The rigorous theorems in Section 3 suggest that solutions of the NLS (2) with $\psi_0(\mathbf{x}) \in H^1$ always exist globally when $p < q$. These theorems, however, do not cover the case $p = q$. The asymptotic analysis and simulations of [7, 20] strongly suggest that in the critical case, when $p = q$ the solution always exists globally, see Continuation Results 1 and 2. Since there are no rigorous and asymptotic results for the supercritical case with $p = q$, we study this case numerically.

Consider the one-dimensional NLS with $p = q$

$$i\psi_t(t, x) + \psi_{xx} + (1 + i\delta)|\psi|^{p-1}\psi = 0, \quad (26a)$$

with the perturbed solitary-wave initial condition

$$\psi_0(x) = 1.05R_p(x), \quad (26b)$$

where $R_p(x)$ is given by (5). In Figure 1 we solve (26) with $\delta = 5 \cdot 10^{-3}$, and plot

$$L(t) = \left| \frac{\psi(0,0)}{\psi(t,0)} \right|^{(p-1)/2}. \quad (27)$$

In the critical case $p = 5$, the collapse is arrested after focusing by ≈ 10 . In the supercritical case $p = 7$, however, the collapse is not arrested after focusing by 10^5 . This and similar simulations suggest that unlike the critical case, *in the supercritical case, the condition $p < q$ is necessary for ensuring global existence in (2).*

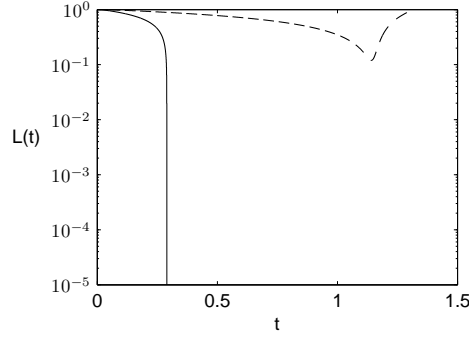


Figure 1: Solution of (26) for $\delta = 5 \cdot 10^{-3}$ with $p = 5$ (dashes) and $p = 7$ (solid).

5. Supercritical NLS

We now consider the effect of small nonlinear damping in the supercritical NLS. Let $\psi^{(\delta)}(t, x)$ be the solution of the one-dimensional supercritical damped NLS ($d = 1$, $p = 7$, $q = 9$)

$$i\psi_t(t, x) + \psi_{xx} + |\psi|^6\psi + i\delta|\psi|^8\psi = 0, \quad (28a)$$

with the initial condition

$$\psi_0(x) = 1.3e^{-x^2}. \quad (28b)$$

Let

$$T_{\max}^{(\delta)} = \arg \max_t \|\psi^{(\delta)}(t, x)\|_{\infty}, \quad (29)$$

denote the time at which the focusing is maximal.

In Figure 2 we plot the solution of (28) for various values of δ . In all cases, the collapse is arrested in a highly asymmetric way with respect to $T_{\text{arrest}}^{(\delta)}$. In addition, the post-collapse defocusing rate of the singular core "appears" to increase to infinity as $\delta \rightarrow 0+$.

In Figure 3 we compare the profile of the solution of (28) with $\delta = 10^{-3}$ with the supercritical ψ_Q and ψ_R profiles, where

$$|\psi_R(t, x)| = \frac{1}{L_R^{2/(p-1)}(t)} R\left(\frac{x}{L_R(t)}\right), \quad L_R(t) = \left| \frac{R(0)}{\psi^{(\delta)}(t, 0)} \right|^{(p-1)/2}, \quad (30a)$$

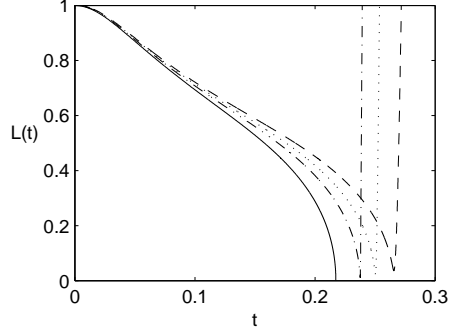


Figure 2: Solution of (28) for $\delta = 0$ (solid), $\delta = 5 \cdot 10^{-3}$ (dashes-dots), $\delta = 7.5 \cdot 10^{-3}$ (dots), and $\delta = 10^{-2}$ (dashes).

$$|\psi_Q(t, x)| = \frac{1}{L_Q^{2/(p-1)}(t)} \left| Q\left(\frac{x}{L_Q(t)}\right) \right|, \quad L_Q(t) = \left| \frac{Q(0)}{\psi^{(\delta)}(t, 0)} \right|^{(p-1)/2}, \quad (30b)$$

and R and Q are the solutions of (4) and (15), respectively, with $d = 1$ and $p = 7$. The NLS solution initially approaches the ψ_Q profile, see Figure 3(a–c). This is to be expected, since when $\delta = 0$ the solution collapses with the ψ_Q profile, see Section 2.2. As the solution approaches $T_{\text{arrest}}^{(\delta)}$, however, the collapsing core moves away from ψ_Q and towards ψ_R , see Figure 3(d), and it remains close to ψ_R for a "short time" after $T_{\text{arrest}}^{(\delta)}$, see Figure 3(e). Eventually, as the collapsing core continues to defocus, it interacts with its tail and "loses" its ψ_R profile, see Figure 3(f).

Next, we repeat the above simulation with a higher nonlinear damping exponent ($q = 11$). Specifically, we solve the NLS

$$i\psi_t(t, x) + \psi_{xx} + |\psi|^6\psi + i\delta|\psi|^{10}\psi = 0, \quad (31a)$$

with the initial condition

$$\psi_0(x) = 1.3e^{-x^2}. \quad (31b)$$

Figure 4 and 5 show that the qualitative behavior of the solution is exactly the same as that of the solution of (28).

Therefore, we conclude that solutions of the supercritical NLS (2) with $q > p > 1 + 4/d$ and $0 < \delta \ll 1$:

1. Exist globally.
2. Are highly asymmetric with respect to $T_{\text{arrest}}^{(\delta)}$.
3. The post-collapse velocity of the defocusing core goes to infinity as $\delta \rightarrow 0+$.
4. The asymptotic profile around $T_{\text{arrest}}^{(\delta)}$ is ψ_R , and not ψ_Q .

The fact that as $t \rightarrow T_{\text{arrest}}^{(\delta)}$ the profile is not given by ψ_Q is not surprising, since the nonlinear damping perturbation obviously has a significant effect near $T_{\text{arrest}}^{(\delta)}$, and therefore there is no reason why it should not change the solution profile. What is surprising is that the profile changes to the supercritical ψ_R profile. As far as we know, this is the first observation in which the asymptotic profile in the supercritical NLS is given by the supercritical ψ_R profile.²

²The standing-ring solutions of the undamped supercritical NLS with $p = 5$ and $d > 1$ also collapse with the ψ_R profile [23, 24, 25, 26]. In that case, however, ψ_R is the asymptotic profile of the *critical* one-dimensional quintic NLS.

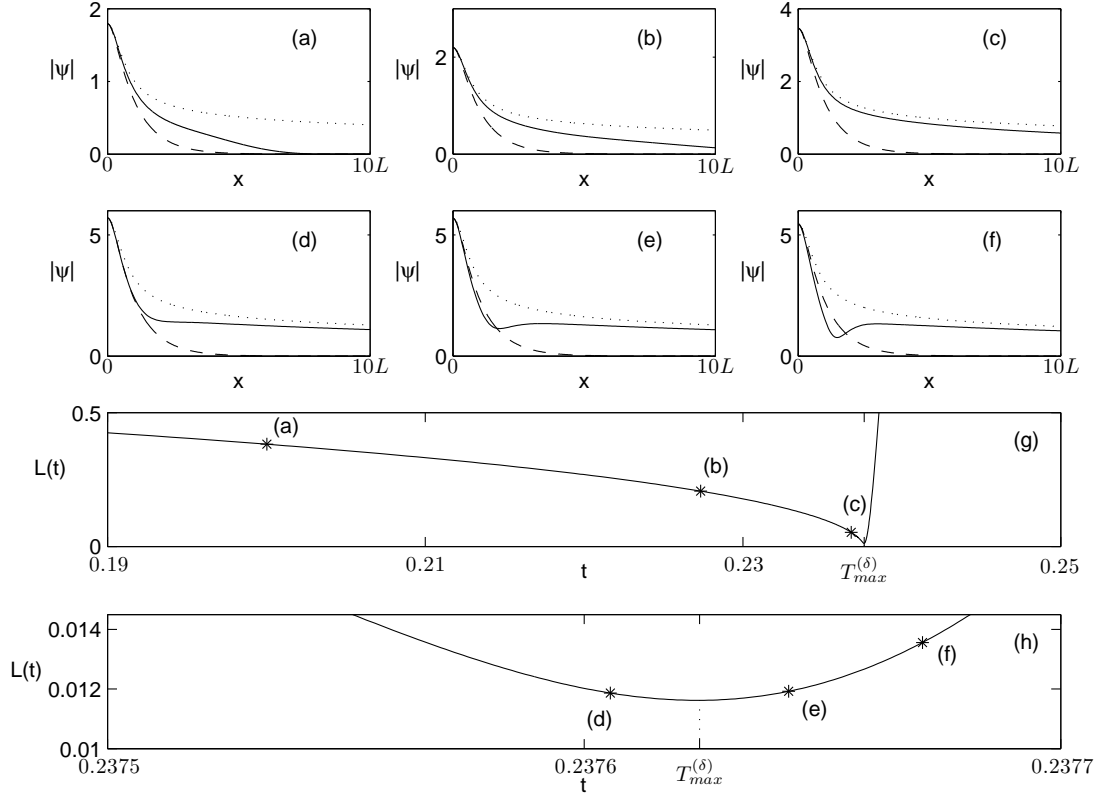


Figure 3: Solution of (28) for $\delta = 5 \cdot 10^{-3}$ (solid), and the fitted $|\psi_Q|$ (dots) and $|\psi_R|$ (dashes). (a) $t \approx 0.2$, $L \approx 0.38$ (b) $t \approx 0.227$, $L \approx 0.2$ (c) $t \approx 0.2368$, $L \approx 0.05$ (d) $t \approx 0.2376$, $L \approx 0.0118$ (e) $t \approx 0.23764$, $L \approx 0.0119$ (f) $t \approx 0.23767$, $L \approx 0.135$. The stars in (g) and (h) denote the values of t and $L(t)$ for the data in subplots (a)–(f).

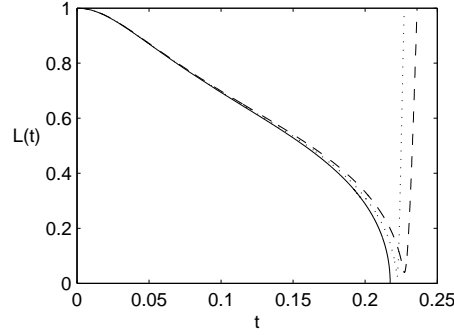


Figure 4: Solution of (31) for $\delta = 0$ (solid), $\delta = 5 \cdot 10^{-4}$ (dots), and $\delta = 10^{-3}$ (dashes).

6. Critical NLS

6.1. Continuation of loglog collapse

In [7], we studied the effect of nonlinear damping in the critical NLS with $p = q$ with initial conditions that lead to a loglog collapse, see Continuation Result 2. We now consider the case $p > q$.

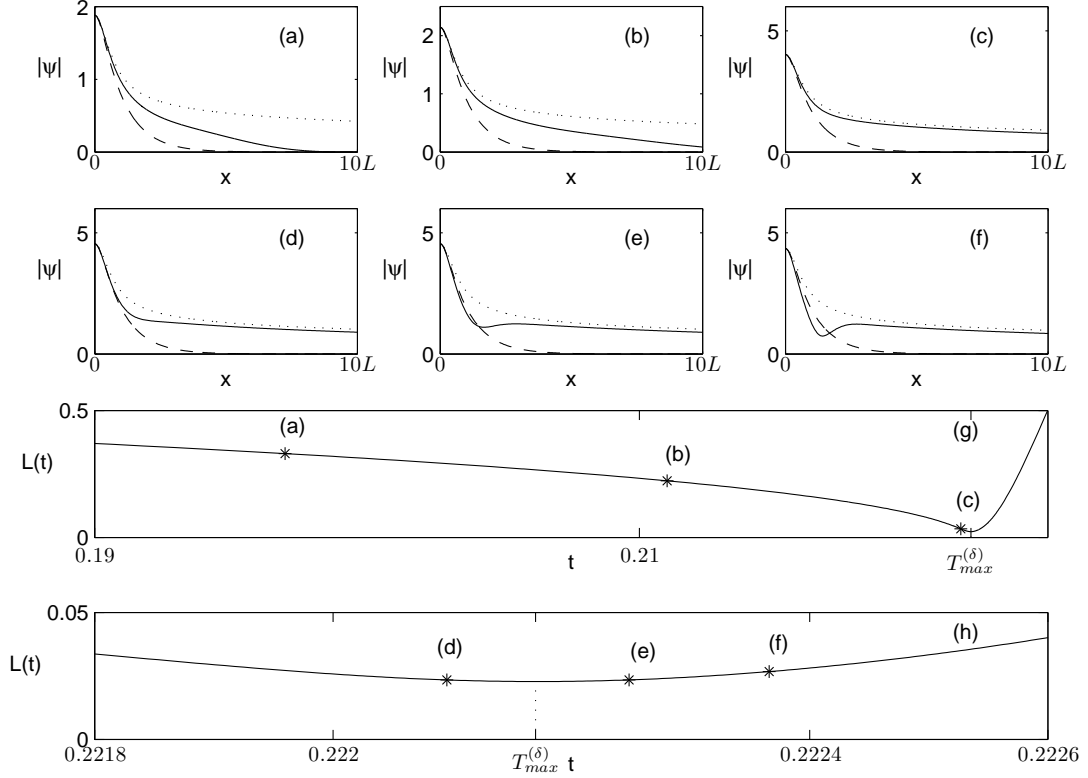


Figure 5: Same as Figure 3 for the solution of (31). (a) $t \approx 0.19$, $L \approx 0.33$ (b) $t \approx 0.211$, $L \approx 0.22$ (c) $t \approx 0.221$, $L \approx 0.033$ (d) $t \approx 0.222$, $L \approx 0.02341$ (e) $t \approx 0.2222$, $L \approx 0.02343$ (f) $t \approx 0.2223$, $L \approx 0.0267$.

Consider the damped one-dimensional critical NLS ($d = 1$, $p = 5$, $q = 7$)

$$i\psi_t(t, x) + \psi_{xx} + |\psi|^4\psi + i\delta|\psi|^6\psi = 0, \quad (32a)$$

with the initial condition

$$\psi_0(x) = 1.6e^{-x^2}, \quad (32b)$$

whose power is 4% above the critical power for collapse. When $\delta = 0$, the NLS solution collapses with the ψ_R profile at the loglog blowup rate.

In Figure 6 we solve (32) for various values of δ . In all cases, the collapse is arrested in a highly asymmetric way with respect to $T_{\text{arrest}}^{(\delta)}$. In addition, the post-collapse defocusing rate appears to increase to infinity as $\delta \rightarrow 0+$. This qualitative behavior is as in the case $p = q$, see Continuation Result 2. Therefore, we conclude that the qualitative behavior for $q = p$ and for $q > p$ is the same.

In Figure 7 we compare the profile of the solution of (32) with $\delta = 10^{-5}$ with the best-fitting critical ψ_R profile, see (30a). The NLS solution initially approaches the ψ_R profile, see Figure 7(a–c). This is to be expected, since when $\delta = 0$ the solution collapses with the ψ_R profile, see Theorem 2. As the solution approaches $T_{\text{arrest}}^{(\delta)}$, however, the collapsing core moves away from ψ_R , see Figure 7(d–e). Unlike the supercritical case, however, the solution profile near $T_{\text{arrest}}^{(\delta)}$ is still “close” to ψ_R . This is because in the critical case, perturbations arrest the collapse when they are still small compared with the nonlinearity and diffraction [27]. Eventually, as the collapsing core

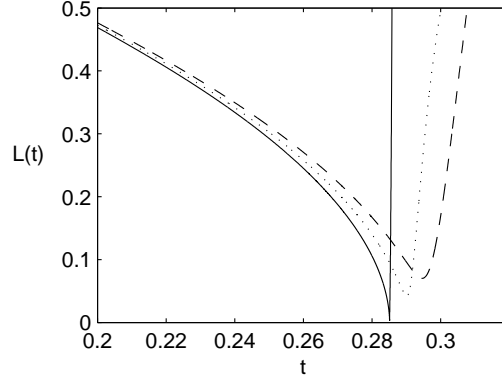


Figure 6: Solution of (32) for $\delta = 10^{-5}$ (solid), $\delta = 2.5 \cdot 10^{-4}$ (dots), and $\delta = 5 \cdot 10^{-4}$ (dashes).

continues to defocus, it interacts with its tail and "loses" its ψ_R profile, see Figure 7(f).

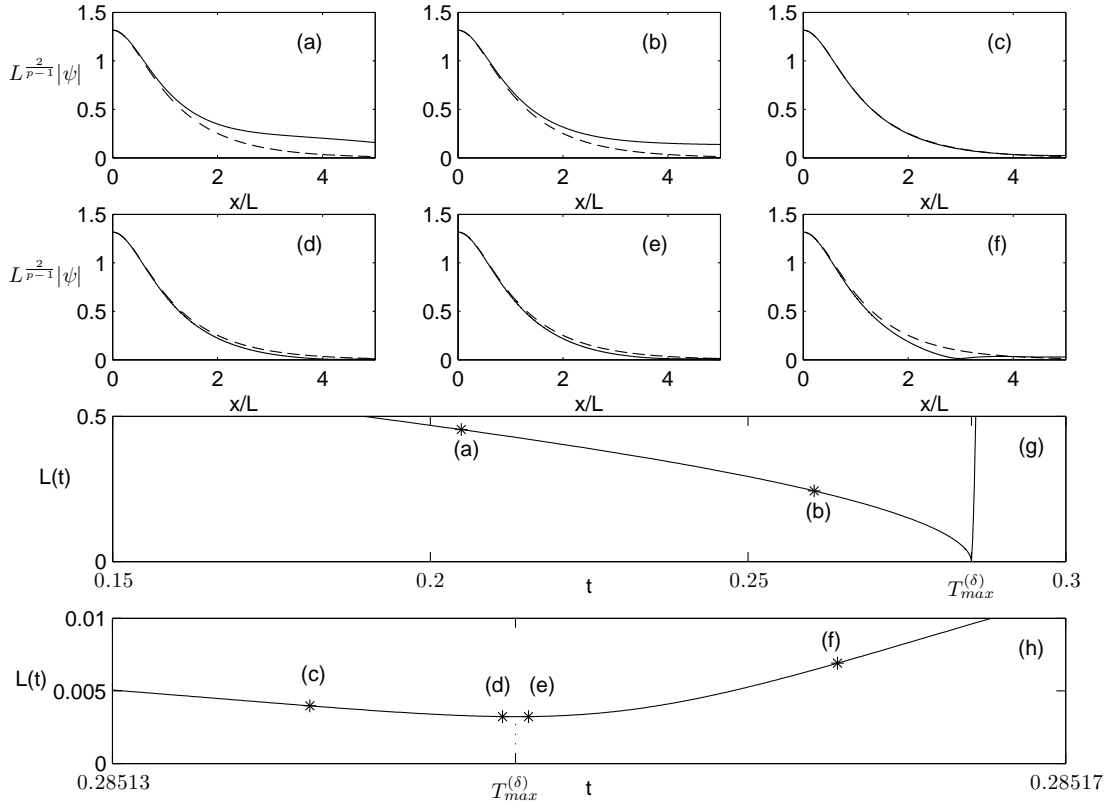


Figure 7: Solution of (32) for $\delta = 10^{-5}$ (solid), and the fitted $|\psi_R|$ (dashes). (a) $t \approx 0.205$, $L \approx 0.45$ (b) $t \approx 0.26$, $L \approx 0.24$ (c) $t \approx 0.28514$, $L \approx 0.0037$ (d) $t \approx 0.285146$, $L \approx 0.0032$ (e) $t \approx 0.285147$, $L \approx 0.0032$ (f) $t \approx 0.28515$, $L \approx 0.0046$. The stars in (g) and (h) denote the values of t and $L(t)$ for the data in subplots (a)–(f).

In summary, nonlinearly-damped loglog solutions of the critical NLS with $q \geq p$ have the following properties:

1. The solutions are highly asymmetric with respect to $T_{\text{arrest}}^{(\delta)}$.
2. The post-collapse defocusing velocity goes to infinity as $\delta \rightarrow 0+$.
3. The asymptotic profile near $T_{\text{arrest}}^{(\delta)}$ is "slightly" different from ψ_R .

6.2. Continuation of ψ_{explicit}

Consider the critical NLS with nonlinear damping

$$i\psi_t(t, \mathbf{x}) + \Delta\psi + |\psi|^{4/d}\psi + i\delta|\psi|^{q-1}\psi = 0, \quad 0 < \delta \ll 1, \quad (33a)$$

and the initial condition

$$\psi_0(r) = \psi_{\text{explicit}}(0, r). \quad (33b)$$

When $\delta = 0$, the solution is given by ψ_{explicit} , see equation (8). In [7], we calculated explicitly the continuation of ψ_{explicit} when $q = p$, see Continuation Result 1. We now consider the continuation for $q \neq p$.³

As in [7], we can use *modulation theory* [27] to approximate equation (33) with a reduced system of ordinary-differential equations.

Lemma 1. *Let*

$$L(t) = \left| \frac{\psi(0, 0)}{\psi(t, 0)} \right|^{2/d},$$

where ψ is the solution of equation (33). Then, as $\delta \rightarrow 0+$, the evolution of $L(t)$ is governed by the reduced equations

$$\beta_t(t) = -\frac{2c_q\delta}{M} \frac{1}{L^{(q-1)d/2}}, \quad L_{tt}(t) = -\frac{\beta(t)}{L^3}, \quad (34a)$$

subject to the initial conditions

$$\beta(0) = 0, \quad L(0) = 0, \quad L_t(0) = -1, \quad (34b)$$

where

$$v(\beta) = \begin{cases} c_\nu e^{-\pi/\sqrt{\beta}}, & \beta > 0, \\ 0, & \beta \leq 0, \end{cases}$$

$$c_\nu = \frac{2A_R^2}{M}, \quad A_R = \lim_{r \rightarrow \infty} e^r r^{(d-1)/2} R^{(0)}(r),$$

$$M = \frac{1}{4} \int_0^\infty r^2 |R^{(0)}|^2 r^{d-1} dr, \quad c_q = \|R^{(0)}\|_{q+1}^{q+1},$$

and $R^{(0)}$ is the ground state of (7).

Proof. In [20] it was shown that the reduced equations for the damped NLS (33a) are given by

$$\beta_t(t) = -\frac{\nu(\beta)}{L^2} - \frac{2c_q\delta}{M} \frac{1}{L^{(q-1)d/2}}, \quad L_{tt} = -\frac{\beta(t)}{L^3}. \quad (35)$$

In addition, the initial conditions for the reduced equations (35) that correspond to the initial condition (33b) are $\beta(0) = 0$, $L(0) = 0$, and $L_t(0) = 0$, see [7]. Since $\beta(0) = 0$, and since $\beta_t < 0$, then $\beta(t) < 0$. Hence, $\nu(\beta) \equiv 0$. Therefore, the reduced equations are given by (34). \square

³Since ψ_{explicit} has exactly the critical power for collapse, any amount of damping will arrest the collapse. Therefore, the continuation of ψ_{explicit} can also be defined for $q < p$.

The reduced equations variable $L(t)$ is the solution width, and is also inversely proportional to the solution amplitude. The reduced equations variable $\beta(t)$ is a measure of the acceleration of $L(t)$, and is also linearly proportional to the excess power above P_{cr} of the collapsing core.

Since *modulation theory* is not rigorous, in Figure 8 we compare the numerical solutions of the reduced equations (34) and the NLS (32). This comparison shows that the two solutions are in excellent agreement, thus providing a strong support to the validity of the reduced equations. Therefore, in what follows we study asymptotically and numerically the limit $\delta \rightarrow 0+$ within the framework of the reduced equations, which is considerably easier than studying the limit $\delta \rightarrow 0+$ of the nonlinearly-damped NLS.

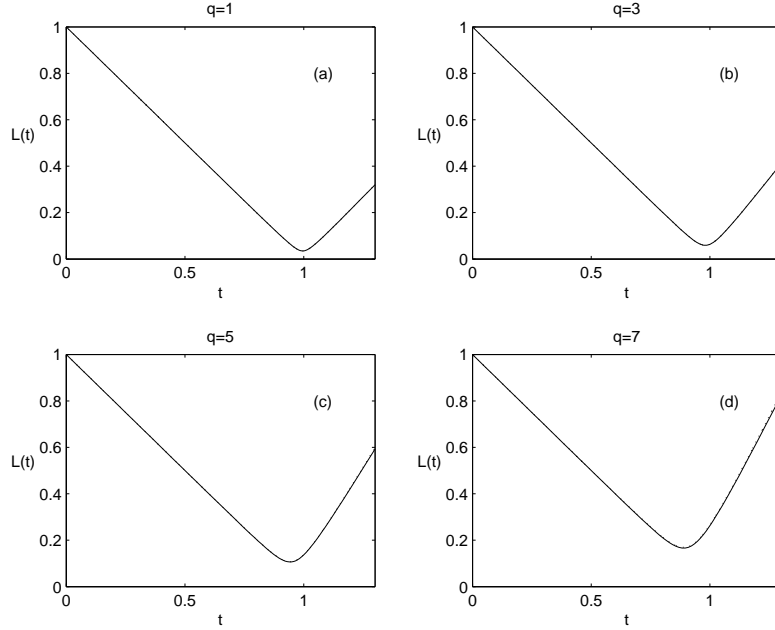


Figure 8: Solution of the reduced equations (34) [solid], and of the NLS (33) [dashes], for $\delta = 2.5 \cdot 10^{-5}$ and $d = 1$. The two curves are indistinguishable. (a) $q = 1$. (b) $q = 3$. (c) $q = 5$. (d) $q = 7$.

The extension of Continuation Result 1 to $q \neq p$ is as follows.

Continuation Result 3. *Let $\psi^{(\delta)}(t, r)$ be the solution of the NLS (33). Then, for any $\theta \in \mathbb{R}$, there exists a sequence $\delta_n \rightarrow 0+$ (depending on θ), such that*

$$\lim_{\delta_n \rightarrow 0+} \psi^{(\delta_n)}(t, r) = \begin{cases} \psi_{\text{explicit}}(t, r) & 0 \leq t < T_c, \\ \psi_{\text{explicit}, \kappa(q)}^*(2T_c - t, r)e^{i\theta} & T_c < t < \infty. \end{cases} \quad (36)$$

In particular, the limiting width of the solution is given by

$$\lim_{\delta \rightarrow 0+} L(t) = \begin{cases} T_c - t & 0 \leq t < T_c, \\ \kappa(q)(t - T_c) & T_c < t < \infty. \end{cases} \quad (37)$$

Proof. We only provide an informal proof, using the reduced equations (34). As $\delta \rightarrow 0+$, $\beta_t(t) \rightarrow 0$, see equation (34a). Therefore, since $\beta(0) = 0$, then $\beta(t) \rightarrow 0$. Hence, $L_{tt}(t) \rightarrow 0$.

Therefore, $\lim_{\delta \rightarrow 0+} L(t)$ is linear in t . Since $\lim_{\delta \rightarrow 0+} L(T_c) = 0$, it follows that

$$\lim_{\delta \rightarrow 0+} L(t; \delta) = \begin{cases} T_c - t, & t < T_c, \\ \kappa(q)(t - T_c), & T_c < t. \end{cases}$$

The loss of phase follows from the fact that $\lim_{t \rightarrow T_c} \arg(\psi_{\text{explicit}}(t, 0)) = \infty$, see [7]. \square

The result of Continuation Result 3 can be explained as follows. By continuity, $\lim_{\delta \rightarrow 0+} \psi^{(\delta)} = \psi_{\text{explicit}}$ for $0 \leq t < T_c$. The limiting solution for $t > T_c$ is an NLS solution that becomes singular as $t \rightarrow T_c+$, and has exactly the critical power at the singularity. Hence, the limiting solution is a minimal-power solution. Therefore, it has to be given by $\psi_{\text{explicit}, \alpha}$ [1, 4].

In Figure 9 we solve the reduced equations (34) with $\delta = 10^{-7}$, and observe that:

1. The limiting solutions are indeed linear for $t < T_c$ and $t > T_c$.
2. The continuation is asymmetric with respect to T_c .
3. The post-collapse slope $\kappa(q)$ increases with q . In [7] we showed that the jump discontinuity in $\lim_{\delta \rightarrow 0+} L_t^2$ at T_c is related to the increase of the Hamiltonian as the limiting solution passes through the singularity. As q increases, damping affects become more pronounced, hence there is a larger increase of the Hamiltonian, hence of the post-collapse slope.
4. When $q = 1$, $\kappa(q = 1) = 1$, i.e., $L(t)$ is symmetric with respect to T_c . Therefore, *the linear damping continuation of ψ_{explicit} is symmetric with respect to T_c , even though the problem is not time-reversible.*

Note that the value of $\kappa(q = 1 + 4/d) \approx 1.614$ was computed analytically in Continuation Result 1.

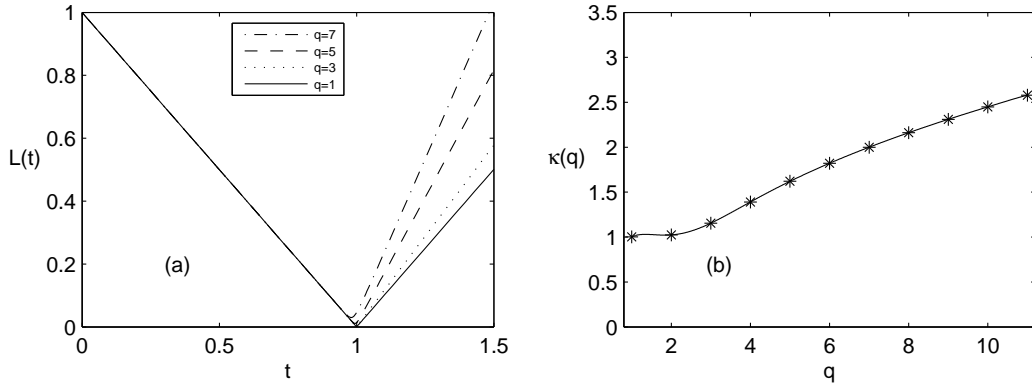


Figure 9: Solution of the reduced equations (34) with $\delta = 10^{-7}$ and various values of q . (a) $L(t)$. (b) $\kappa(q)$.

7. Final remarks

In this study we used numerical simulations to study the effect of small nonlinear-damping on singular NLS solutions. These simulations suggest that the effect of small nonlinear damping is qualitatively the same in the critical NLS with generic initial conditions that lead to a loglog collapse with the ψ_R profile, and in the supercritical NLS with generic initial conditions that lead to a square-root collapse with the ψ_Q profile. Moreover, the qualitative effect of nonlinear damping is independent of the value of q , so long as $q > p$ in the supercritical case and $q \geq p$ in the critical

case. Thus, because nonlinear damping destroys the NLS time reversibility, the nonlinearly-damped solution is highly asymmetric with respect to the arrest-of-collapse time $T_{\text{arrest}}^{(\delta)}$. The post-collapse defocusing velocity $L_t(t)$ of the singular core goes to infinity as $\delta \rightarrow 0+$, since the focusing velocity before the singularity goes to infinity for loglog and square-root blowup rates, and since nonlinear damping increases the Hamiltonian,⁴ hence the “kinetic energy”.

Around $T_{\text{arrest}}^{(\delta)}$, the collapsing core of the singular core moves away from the asymptotic profile of the undamped solution. In the supercritical case the difference between the solution profile and ψ_Q for $t \approx T_{\text{arrest}}^{(\delta)}$ is large. This is intuitive, since damping effects have a large effect when they arrest the collapse. In the critical case, however, the difference between the solution profile and ψ_R for $t \approx T_{\text{arrest}}^{(\delta)}$ is minor. This is because critical collapse has the unique property that it can be arrested by small perturbations [27].

Surprisingly, in the supercritical case the profile of the nonlinearly-damped solution near $T_{\text{arrest}}^{(\delta)}$ appears to be given by the supercritical ψ_R profile. To the best of our knowledge, this is the first observation of a solution of the supercritical NLS that approaches the supercritical ψ_R profile.

Acknowledgment

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⁴If we multiply the NLS (2) by ψ_t^* , add the complex-conjugate equation, and integrate by parts, we get that

$$H_t = i\delta \int |\psi|^{q-1} \psi \psi_t^* + c.c.,$$

where *c.c.* stand for complex conjugate. Let $\psi = Ae^{iS}$, where A and S are real, Then,

$$H_t = 2\delta \int |A|^{q+1} S_t.$$

Since for collapsing solutions $S_t \sim L^{-2}(t)$, it follows that $H_t > 0$.

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